Time series analysis for crisis times: Searching alternatives for standard risk models

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Accepted 21 March, 2014

Abstract

Kim et al. (2011) stated that it appears to be a consensus that the recent instability in global financial markets may be attributable in part to the failure of financial modeling: standard risk models have failed to properly assess the risks associated with large adverse stock price behavior. Their empirical evidences indicate that time series modeled with a framework of stable and tempered stable innovations show better predictive power in measuring market risk compared to standard models based on the normal / t-student distribution assumption. This approach was tested for a Brazilian case. The MLE (Maximum Likelihood Estimation) tests – applied do the daily returns of de closing prices of the IBOVESPA stock exchange index, from January, 1999 to April, 2012, covering some high volatility periods – show that GARCH models that adopt Alpha-Stable or Tempered Stable distributions are a better fit for financial time series that cover turbulent periods. The results suggest an alternative for the Basel III Stressed Var approach.

Keywords: Equity Markets; Stable Distributions; GARCH Models; BASEL III; Value at Risk.

INTRODUCTION

The forecast activity of financial instruments is essential to risk management and portfolio allocation. The debate between the financial sector and its regulators comprehend the evaluation if the sophisticated mathematical and statistical tools that have been used for risk management and the evaluation of complex financial instruments played a crucial role in the recent global financial crisis. The risk models such as the value at risk (VaR) and “black box” models used by the institutional investors and regulated financial entities have been blamed for the bad forecast of the financial assets - see Turner (2009) and Sheedy (2009).This is the context to discuss a model of distribution of returns that is able to explain the high volatility periods.

The volatility cluster effect may be captured by the heteroscedastic autoregressive conditional models (ARCH) and the generalized ARCH models (GARCH) and models formulated by Engle (1982) and Bollerslev (1986). However, there are numberless empirical evidences showing that GARCH models based on the
Table 1. Recent crisis and the S&P 500 daily returns

<table>
<thead>
<tr>
<th>Crash</th>
<th>Day</th>
<th>Returns (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black Monday</td>
<td>19/10/1987</td>
<td>-23</td>
</tr>
<tr>
<td>Asian Crisis</td>
<td>27/10/1997</td>
<td>-7</td>
</tr>
<tr>
<td>Russian Collapse</td>
<td>31/10/1998</td>
<td>-7</td>
</tr>
<tr>
<td>Dot.com Crisis</td>
<td>14/04/2000</td>
<td>-6</td>
</tr>
<tr>
<td>Global 2007 Crisis</td>
<td>29/09/2009</td>
<td>-9</td>
</tr>
</tbody>
</table>


normal – or even t-student - distribution do not perform well in turbulent times. Some examples were given by Kim et al (2011), for the S&P/500 stock exchange index, in periods of time that included the black Monday crisis (October 19, 1987), and the recent global economic crisis, this later linked to events such as the mortgages subprime collapse in 2007 and the Lehman Brothers failure in September, 2008.

Other several studies show that revenues from financial assets have well defined leptokurtosis and asymmetry patterns that cannot be captured by the assumption of normality. The non-normality assumption was recently regarded by Sorwar and Dowd (2010), Fajardo and Farias (2010), and by Bedendo et al (2010), to investigate the options pricing verification models. The extreme values theory (EVT) was applied to measure the financial risk and to capture the extreme events that cannot be described by the normal distribution, as stated by Neftci (2000), Bali (2003) and Gupta and Liang (2005): the EVT approach cannot be applied in a no-arbitrage scenario, since extreme values distributions and generalized distributions by Pareto (as the t-Student distributions) do not lead to semi-martingale procedures and it is impossible to find a martingale measure equivalent for options pricing.

Introducing the alpha-stable and the Tempered stable models

Kim et al. (2011), evaluated alpha-stable and Tempered Stable (TS) models and found out that there is a superior prediction power in relation to other models that are widely used by the industries. Those models were applied for the daily returns series of the S&P 500 during highly volatile periods. The empirical evidences indicated that time series models with alpha-stable or tempered stable innovation have better predictive power of the market risk in comparison to the standard models based on the normal distribution assumption. Instead, time series models based on normal innovation do not offer a reliable prediction even if they represent volatility groupings. Time series models based on t-Student innovations were also empirically refused.

Var Approach

Due to the tail properties of the TS distributions, Kim et al. (2011) showed that the quantity of necessary capital by a risk manager who uses a TS-based model is smaller than the one used by other models. In contrast with t-student or EVT innovation models, the CTS (Classical Tempered Stable) assumption allows a good market model for prices options. Kim et al (2011) chose five recent turbulent periods: 1987(Black Monday), 1997(Asian Tigers’ crisis), 1998(Russian Default), 2000(Dotcom Collapse) and 2008(USA Financial Crisis). Then, analyzed the VaR approach based on different distribution assumptions and performed the VaR backtest considering the latest four year- log-returns for the S&P/500. Based on the likelihood reason test by Christoffersen (1998), the normally distributed models were refused. Table 1.

Normal Distributions x Reality

Considering the results and the last table, the probability of a new Black Monday, based on the three normal models from the related article, is considerably low compared with the proposed stable models. Conversely, the average times of occurrences based on the three normal models is very high. Each crash occur every 7,966 x10^{141} years in the normal model of constant volatility, 2,554x10^{39} years in the GARCH normal density model and 2,904x10^{39} years in the ARMA-GARCH normal density model. These values are bigger than the age of the Universe. In real life, this fall would be expected at each “30 to 50 year – interval”. The authors conclude that models based on normal distributions are not realistic.

Replicating for a Brazilian Case

The Kim’s et al approach (2011) was applied to the daily returns series of the closing prices of the IBOVESPA Brazilian stock exchange Index - from January, 1999 to April, 2012 - verifying if it fits best to alpha-stable and/or
Classical Tempered Stable densities in comparison with normal (and t-student) densities. The chosen time frame period begins when the Brazilian currency started to float. Several unconditional volatility levels can be explained by several polit-economic scenarios. Among them, some high volatility periods where the mid-2002 confidence crisis in the eve of the Presidential Elections (October 27, 2002) when Luís Inácio da Silva (Lula) was elected for his first term as Brazilian President; and the international financial crisis that began in 2007, specially in 2008 (a high 5.81% unconditional volatility level, from September 8, 2008 to November 21, 2008).

Base III

From 2007 onwards the world is going through troubled times, unchained by the international financial crisis that motivated the so called new Basel Agreement – Basel III, released from the BIS (Bank for International Settlements.

Basel III, in the words of BIS General Manager by 2010 – see Caruana (2010) - has resulted in significant progress in prudential financial regulation since the beginning of the global financial crisis. In other words, there is a new global economic context that imposes major challenges, such as the "subprime mortgage crisis". The Basel III proposal contains some items that are a radical revision of Basel II. In another perspective, Basel III is not a new agreement, but rather a set of proposed amendments to the Basel II agreement, changing the latter measures that were deemed insufficient, either in conception, or in the used metric. In general, the Basel III either increases the requirements in Basel II or creates new demands, where the crisis has highlighted the procedures to be insufficient to control the instability of the financial markets or to avoid occurrence of more serious crises.

Diffusion problems and leaps

A way to deal with diffusion problem is the use of Semimartingales, but the structure of these procedures is very complex. The alternative is the use of Lévy, additive processes (non homogeneous processes) or the use of models of stochastic volatility with leaps (Ornstein-Uhlenbeck). Following Kim’s et al approach (2011), a distribution based on Lévy’s processes is tested, which allows the modeling without resorting to much abstraction.

Mandelbrot (1963) was pioneer on the use of stable distributions (or alpha-stable) to model skewness distributions and fat tails. In a general sense, the alpha-stable distributions family is a varied class which includes as subclasses the following distributions: the Gaussian, Cauchy’s and Lévy’s distribution (or Gaussian or Pearson V). The Lévy’s continuous stochastic procedure has stationary and independent increments.

Coming sections

The coming sections comprehend a further bibliographical review (section 2) - introducing the alpha-stable and tempered stable distributions - the methodologies (section 3) and the results (section 4). The conclusion section (section 5) has a paragraph suggesting a possible alternative to the BIS (Bank for International Settlements) recommendation for risk modeling, in a recent Basel III Agreement Document (2011).

Further Review

It is clear from previous works, like Valls and Almeida (2000), that the chosen series - the daily returns of the closing prices of the IBOVESPA Stock Exchange Index - presents volatility clusters and ARCH effects. For a review of the ARCH and GARCH effects, Engle (1982) and Bollerslev (1986) are the starting points. The existence of volatility clusters suggests either an approach under the viewpoint of changes in volatility regimes or an approach under the standpoint of volatility leaps. For the changes in volatility regimes, see Hamilton (1990) with the SW (Switching Regimes) models review. Referring to leaps, a quick review of the Lévy processes literature started from the Kim et al. (2011) review. A previous identification of the unconditional volatility levels is considered, following Inclán and Tiao (1994).
general and robust class of tempered-stable distributions and an identifiable parameterization according to Rachev et al. (2007), Scherer et al. (2009) and Kim et al. (2011). These representations exhibit alpha-stable and gaussian trends in tempered stable procedures, therefore they offer a probabilistic trait and they may also be used for simulations.

Changes of volatility regimes
In order to verify if there are levels of unconditional volatility, the determination of the discrete changes in the unconditional variance may be made with the ICSS-Iterative Cumulative Sum of Squares – the Inclán and Tiao (1994) algorithm.

Switching regimes
A great part of the bibliography that deals with regime changes (mean and/or volatility) has Hamilton (1990) – and other of his works - as a main source. Hamilton utilizes the EM algorithm to obtain the maximum likelihood estimation (MLE) of the procedures parameters subject to discrete changes in their self-regression parameters. The premise of the model is that many of the movements in the assets prices appear from specific identifiable events. The author considers parameters such as level, variance regression or the proper dynamics of a self-regression - being subject to occasional and discrete changes. The probability law that governs such changes is openly declared and it is supposed that these changes exhibit a proper dynamic conduct. Cai (1994) conclusions recommends SWGARCH models in place of SWARCH models.

The SWGARCH models combine GARCH with regime changes. Based on the models from Haas et al (2004), it was possible to offer a direct estimate of the maximum likelihood function.

Stable distributions
The alpha-stable algorithms source code came from Veillete (2010). An alpha-stable distribution is calculated: to obtain initial parameters, using the STBLFIT function.

Equation 3: Characteristic function - CTS
\[
\Phi(u;\alpha;C+;\lambda+;\lambda-;m) = \exp\left(iu\gamma(1-\alpha)(C+\lambda+\alpha-1-C-\lambda-\alpha)\right), \quad \gamma = \mathcal{A} - (A+iu)\alpha - \lambda \alpha \quad \text{and} \quad A3 = C\Gamma(-\alpha)((\lambda+iu)\alpha - \lambda - \alpha)
\]

METHODOLOGY
Methodology for identifying sudden changes in Variance
The determination of the discrete changes in the unconditional variance is identified with the ICSS (Interactive Cumulative Sum of Squares) algorithm, developed by Inclán and Tiao (1994).

GARCH and SWGARCH Models
The SWGARCH models combine GARCH with regime changes. Based on the models from Haas et al (2004), it was possible to offer a direct estimate of the maximum likelihood function.

Stable distributions
The alpha-stable algorithms source code came from Veillete (2010). An alpha-stable distribution is calculated: to obtain initial parameters, using the STBLFIT function.

Equation 1 Returns for a SW GARCH model, from Valls (2000) and Haas (2004)
\[
y_t = \mu_t + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_t^2); \quad \sigma_t = \sqrt{\alpha_0 + \sum_{i=1}^{q} \alpha_i r_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} + \xi d_{t-1} r_{t-j}^2}
\]

Where \( \mu_t = \sqrt{g_{\alpha} r_t} \) and \( r_t = \sqrt{h_t \epsilon_t} \)

Equation 2: Variance equation for SWGARCH (k,p,q)-L models, k for level and L for Leverage
\[
h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i r_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} + \xi d_{t-1} r_{t-j}^2;
\]

Where \( d_t - 1 \) stands for the leverage effect: \( d_t - 1 = 1 \) if \( u_t \leq 0 \) and \( d_t - 1 = 0 \), otherwise.

Published by Basic Research Journal of Business Management and Accounts
Setting $C=C_{\alpha}C_\lambda=(\Gamma(2-\alpha)/(\lambda^\alpha + \lambda^{\alpha-2}))^{-1}$, a standard CTS distribution is generated with zero mean and standard deviation equal to one, as adopted by Kim et al (ibid).

### Choice of the algorithms

The algorithms where obtained and adapted from five sources, summarized in Table 2.

### Tests

The time series are the closing prices of the IBOVESPA stock exchange daily index, obtained from the time series webpage of Banco Central do Brasil, from January 4, 1999 to April 27, 2012. The series presents heteroscedasticity, confirmed by the Engle’s test. The IBOVESPA daily returns of the daily closing prices are given by:

$$r_t = \ln(IBOVESPA_t / IBOVESPA_{t-1})$$

The next table and figure illustrate the results from the ICSS algorithm, identifying the levels of unconditional volatility Table 3 and figure 1.

The ratio among volatility levels is significant. The ratio between the lowest level (1.11%, between 06/30/2010 to 07/29/2011) and the highest level (5.81%, between 09/08/2008 and 11/21/2008) is 5.24. This ratio justifies modeling with at least two unconditional volatility levels.

### Initial tests with SWGARCH

Initially, the IBOVESPA series was tested with an AR Model (1) –SWGARCH (1,1) with normal density and two and three volatility levels, optimizing with the MLE method (Maximum Likelihood Estimation) in a Monte Carlo simulation, testing one million parameter

<table>
<thead>
<tr>
<th>Period</th>
<th>Begin</th>
<th>End</th>
<th>Level</th>
<th>Working Days</th>
<th>High Level Volatility: Possible Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>01/04/99</td>
<td>01/29/99</td>
<td>8.25%</td>
<td>25</td>
<td>Brazilian Currency Crisis’s</td>
</tr>
<tr>
<td>2</td>
<td>02/01/99</td>
<td>05/26/99</td>
<td>2.40%</td>
<td>114</td>
<td>Elections eve of President’s Lula first Mandate</td>
</tr>
<tr>
<td>3</td>
<td>05/27/99</td>
<td>12/31/99</td>
<td>1.53%</td>
<td>218</td>
<td>First peak of International Financial Crisis: Lehman Brothers Bankruptcy</td>
</tr>
<tr>
<td>4</td>
<td>01/03/00</td>
<td>06/04/02</td>
<td>2.02%</td>
<td>883</td>
<td>Worsening of the crisis and Basic Interest Rate reduction Eve</td>
</tr>
<tr>
<td>5</td>
<td>06/05/02</td>
<td>10/29/02</td>
<td>2.59%</td>
<td>146</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10/30/02</td>
<td>07/19/06</td>
<td>1.68%</td>
<td>1358</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>07/20/06</td>
<td>07/20/07</td>
<td>1.34%</td>
<td>365</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>07/23/07</td>
<td>09/05/08</td>
<td>2.01%</td>
<td>410</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>09/08/08</td>
<td>11/21/08</td>
<td>5.81%</td>
<td>74</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>11/24/08</td>
<td>05/15/09</td>
<td>2.60%</td>
<td>172</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>05/18/09</td>
<td>06/29/10</td>
<td>1.51%</td>
<td>407</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>06/30/10</td>
<td>07/29/11</td>
<td>1.11%</td>
<td>394</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>08/01/11</td>
<td>08/10/11</td>
<td>4.53%</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>08/11/11</td>
<td>10/31/11</td>
<td>2.04%</td>
<td>81</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>11/01/11</td>
<td>04/26/12</td>
<td>1.27%</td>
<td>177</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2. Algorithms utilized/developed for this text

<table>
<thead>
<tr>
<th>OPERATION</th>
<th>ALGORITHM</th>
<th>SOURCE TEXT/SOURCE CODE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconditional Volatility Level Change</td>
<td>ICSS</td>
<td>Inclán and Tiao (1994)/ modified.</td>
</tr>
</tbody>
</table>
combinations. Testing with two volatility levels resulted in significant values for MLE, contrary to tests with three levels - apart from lacking parsimony.

**Alpha-stable distribution**

The function STBLFIT was applied, where $\alpha$, $\beta$, $\delta$ and $\gamma$ stand for the characteristic parameter (tail), skewness, scale (equivalent to variance) and location (equivalent to mean). When fitting a normal distribution, the tail value is 2, the skewness value is zero, the scale is 1 and the mean is zero.

**Testing with GARCH and SWGARCH**

Some innovations models were tested, and the results for five models are reported: ARMA (1, 1)-SWGARCH (1, 1) with normal density, ARMA (1, 1)-GARCH (1, 1) with normal alpha-stable density, ARMA (1, 1)-SWGARCH (1, 1) with CTS density, ARMA (1, 1)-GARCH (1, 1) with alpha-stable density and ARMA (1, 1)-GARCH (1, 1) with CTS. Each round of simulations comprehended around five thousand Monte Carlo Simulations. Each simulation round lasted about 35 hours, since the probability density of stable distributions was calculated from its characteristic function at each random parameter generation. In this text, three of the simulation rounds are reported

**SWGARCH Tests: First round of simulations**

In the first round, the best models were the ARMA (1, 1)-GARCH (1, 1)-CTS model and the ARMA (1,1)-SWGARCH-CTS model, with almost the same result (MLE=23,118).

**SWGARCH Tests: Second Round of Simulations**

The second round consisted of another group of simulations, only with the stable models. $V_1$ and $V_2$ are the volatility levels, and $P_1$ and $P_2$ are the probabilities to stay in the same volatility level among consecutive observations; $G_1$, $G_2$, $AC_1$, $AC_2$ stands for the GARCH/ARCH parameters; SWG STABLE, SWG CTS, GARCH STBL and GARCH CTS stands for the Log Likelihood of SWGARCH for alpha-stable and CTS distributions. The best model was the ARMA (1,1)-GARCH (1,1)-CTS model, slightly superior to the ARMA (1,1)-SWGARCH-CTS model table 4.

**SWGARCH Tests: Third round of simulations**

In this round of simulations, the transition probability between two markovian states is equal to 50% and also the ARMA (1,1) parameters are fixed. The best model was the ARMA (1,1)-GARCH (1,1)-CTS model, slightly
superior to the ARMA(1,1)-SWGARCH-CTS model. The ARMA parameters are (0.0006641; 0.72791; 0.7533); Vl and V2 stands for the unconditional volatility levels; SWGARCH_STBL, SWGARCH_NORMAL, SWGARCH_CTS, GARCH_STBL and GARCH_CTS stands for Log Likelihood for SWGARCH with alpha-Stable, normal or CTS distributions and Log Likelihood for GARCH alpha-Stable or CTS distributions table 5.

CONCLUSIONS

The recent instability in global financial markets may be attributable in part to the failure of financial modeling. One of the possible reasons for the weak modeling performance was the assumption of normally or t-student distributed innovations. The considered alternative is the stable innovation approach from Kim et al (2011), combined with SWGARCH models, from Haas et al. (2004).

When analyzing the daily return series of the daily closing prices from the main Brazilian stock exchange index - the IBOVESPA stock index, from January,1999 to April,2012 - the normal innovation was compared with alpha-stable and CTS (Classical Tempered Stable) innovations in a ARMA (1, 1) –GARCH (1, 1) and ARMA (1, 1) – SWGARCH (1, 1) models. The highest MLE result occurred with ARMA(1,1)-GARCH(1,1) models with alpha-stable density and ARMA(1,1)-GARCH(1,1)-CTS(Classical Tempered Stable), even superior to the alternatives that adopted two volatility levels, i.e. ARMA(1,1)-SWGARCH(1,1) alpha-stable and ARMA(1,1)-SWGARCH(1,1)-CTS.

The results replicates Kim et al (ibid.) results: financial time series that covers turbulent periods are better modeled with stable innovations, yet Monte Carlo optimizations are considerably more time consuming than equivalent simulations that evaluates only normal or t-student innovations, as stable densities need to be generated from the characteristic functions.

Another disclaimer is verified when creating a time series of daily returns with elements randomly generated by the Monte Carlo Method: the highest MLE is found ed with the ARMA (1,1)-SWGARCH-alpha-stable and not with the ARMA(1,1)-SWGARCH-CTS stable distribution, differently from Kim et al (ibid.). The results suggest that alpha stable distributions can be a best fit for less turbulent times.

On the other hand, and apart from the Kim et al. (ibid) approach, the results show that the switching regime structure do not necessarily add a better fit, suggesting that tempered stable densities endogenize volatility level changes.

Banking regulation and possible alternatives for risk management

The Basel Agreement III (BIS, 2011) - announced in response to the latest international financial crisis - recommendations suggest that quite different specifications for market risk models can be found in the literature, even though the VaR (Value at Risk) approach prevails as the main alternative. BIS III introduce s a stressed value-at-risk - SVaR, a mix of VaR and Stress test - capital requirement based on a continuous 12-month period of significant financial Stress, but keep working with the standard 99% confidence interval (one-tailed), 10-day holding period and the normal density. Instead, Kim et al. (ibid) recommend the Average Value at Risk (AVaR, a.k.a. CVaR) with stable densities in place of the usual normal density, as an alternative to the BIS recommended Stressed VaR approach: the stressed
periods are endogenous to the stable densities. As Pengelly (2011) stated, the new BIS capital charge based on SVaR is blamed for being intellectually inconsistent and could potentially be troublesome for banks to implement. Besides, it is uncertain what guarantees that any significant empirical examples can exhibit a “12-month period of significant financial Stress”. In our example, the longest stress period has 146 working days: the elections eve of 2002.

In a third perspective, the infrastructure of some markets, like the domestic derivatives market in Brazil, are in a stage that all operations must be registered in registration entities duly authorized, forcing some daily limits of oscillation to be specified, as seen in BM&F Bovespa (2013). In other words, the stress values can be directly deduced from the daily limits of oscillation, dispensing the need to search for historical stressed values to compose a stressed VaR.

REFERENCES

Annex

Calculating CTS density

**Equation 2:** Equations for FFT (Fast Fourier Transform)

| \( u_k := -a + \frac{2a}{N} (k - 1) \) | \( u_k^* := \frac{u_{k+1} + u_k}{2} \) |
| \( x_j := -\frac{N\pi}{2a} + \frac{\pi}{a} (j - 1) \) | \( C_j := -\frac{a}{N\pi} (-1)^{j-1} e^{-\pi(j-1)}N \) |


**Middle Point:**

**Equation 3:** Middle point for CTS density

| \( f^{MP}(x_j) = \frac{1}{2\pi} \left[ \int_{-\infty}^{+\infty} e^{-iux_j} \phi(u) du \right] \) | \( f^{MP}(x_j) = \frac{1}{2\pi} \left[ \int_{-\infty}^{+\infty} e^{-iux_j} \phi(u) du \right] \) |
| \( \approx C_j \sum_{k=1}^{N} (-1)^{k-1} \phi(u_k^*) e^{-\frac{2\pi(j-1)(k-1)}{N}} \) | \( \approx C_j FFT_{j\{((-1)^{k-1} \phi(u_k^*)}_{k=1,...,N} \) |

Getting the last point (LP):

| \( f^{LP}(x_j) \approx D_j \sum_{k=1}^{N} (-1)^{k-1} \phi(u_k^*) e^{-\frac{2\pi(j-1)(k-1)}{N}} \) | \( f^{LP}(x_j) \approx D_j \sum_{k=1}^{N} (-1)^{k-1} \phi(u_k^*) e^{-\frac{2\pi(j-1)(k-1)}{N}} \) |
| \( = D_j FFT_{j\{((-1)^{k-1} \phi(u_k^*)}_{k=1,...,N} \) |

Rule of Simpson, from Menn and Rachev(2006):

\[
f(x_j) = \frac{2}{3} f^{MP}(x_j) + \frac{1}{3} f^{LP}(x_j)
\]