Common fixed point theorems in fuzzy menger probabilistic quasi-metric spaces

Ruchi Singh\textsuperscript{1}, AD Singh\textsuperscript{2} and Anil Goyal\textsuperscript{3}

\textsuperscript{1,2}MVM Govt Science College, Bhopal.
\textsuperscript{3}UIT-RGPV, Bhopal.

*Corresponding author email: ruchisingh0107@gmail.com

Accepted 05 September, 2015

ABSTRACT

In 1989, Kent and Richardson [Ordered probabilistic metric spaces, J. Austral. Math. Soc. Ser. A 46(1) (1989), 88-99, MR0966286 (90b:54022)] introduced the class of probabilistic quasi-metric spaces which offers a wider framework than that of metric spaces. The aim of this paper is to prove common fixed point theorems for weakly compatible mappings in Fuzzy Menger probabilistic quasi-metric spaces.

Keywords: Weakly compatible mappings, Fuzzy Menger PQM-space, weakly compatible mapping, Hadzic-type t-norm, common fixed point.

INTRODUCTION

Menger (1942) introduced the notion of a probabilistic metric space (shortly, PM space) in 1942. The study of this space received much attention after the pioneering work of Schweizer and Sklar (1983) (also see Chang et al., 2001). In 1989, Kent and Richardson (1989) introduced and studied the class of probabilistic quasi-metric spaces (shortly, PQM-spaces) and proved common fixed point theorems. Many mathematicians weakened the notion of commutativity by introducing the notions of weak commutativity (Sessa, 1982), compatibility (Jungck, 1986) and weak compatibility (Jungck and Rhoades, 1998) in metric spaces and proved a number of fixed point theorems using these notions. It is worth to mention that each pair of commuting self mappings is weakly commuting, each pair of weakly commuting self mappings is compatible and each pair of compatible self mappings is weak compatible but the converse is not always true. Many authors formulated the definitions of weakly commuting (Singh and Pant, 1985/1986), compatible (Mishra, 1991) and weakly compatible mappings (Singh and Jain, 2005) in probabilistic settings and proved a number of fixed point theorems. Shrivastav et al. (2012), proved fixed point result in fuzzy probabilistic metric space. Fixed point theorems for single-valued mappings have appeared in PQM-spaces (Chauhan, 2010; Mihe, 2008, Mihe, 2009; Mohamad, 2006; Rezaiyan et al., 2008; Pant and Chauhan, 2010; Sastry et al., 2010; Sedghi et al., 2009; Shabani and Ghasempour, 2008). Shrivastav et al. (2012), have given the definition of fuzzy probabilistic metric space and proved fixed point theorem for such space. Recently, Cho (2010) proved common fixed point theorems for set-valued mappings in quasi-metric spaces. The theory of quasi metric spaces can be used as an efficient tool to solve so many several problems like theoretical computer science, approximation theory and topological algebra (Grabiec et al., 2006; Kùnzi, 2001; Mohamad, 2006).

In the paper, we prove common fixed point theorems for weakly compatible mappings in Fuzzy Menger PQM-spaces.
Preliminary Notes

Let us define and recall some definitions:

Definition

(Schweizer and Sklar, 1983). A mapping \( T: [0,1] \times [0,1] \to [0,1] \) is the t-norm if it is satisfying the following conditions:

(a) \( T \) is commutative and associative;
(b) \( T(a,1) = a \forall a \in [0,1] \);
(c) \( T(c,d) \geq T(a,b) \) for \( c \geq a, d \geq b \) and for all \( a, b, c, d \in [0,1] \).

Definition

(Schweizer and Sklar, 1983). A mapping \( F_a: \mathbb{R} \to \mathbb{R}^+ \) is called a fuzzy distribution function for all \( \alpha \in [0,1] \) if it is non-decreasing and left continuous with \( \inf \{ F_a(t) : t \in \mathbb{R} \} = 0 \) and \( \sup \{ F_a(t) : t \in \mathbb{R} \} = 1 \).

We shall denote by \( F \) the set of all fuzzy distribution function defined on \([\mathbb{R}], \mathbb{R}^+\) while \( \varepsilon \) will always denote the specific distribution function defined by

\[ \varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases} \]

If \( X \) is a non-empty set and a mapping \( F_a \) from \( X \times X \) into the collections of all fuzzy distribution functions \( F_a \in R \) for all \( \alpha \in [0,1] \). For \( x, y \in X \) we denote the fuzzy distribution function \( F_a \) at \( (x,y) \) by \( F_a(x,y) \) and \( F_a(x,y) \) is the value of \( F_a(x,y) \) at \( t \) in \( R \).

Definition

(Kent and Richardson, 1989). A Fuzzy Menger PQM-space is a triplet \( (X,F_a,T) \), where \( X \) is a non-empty set, \( T \) is a continuous t-norm and \( F_a \) is a fuzzy probabilistic distance function satisfy the following conditions : for all \( \alpha \in [0,1] \) and \( x, y, z \in X \) and \( s, t \geq 0 \) assumed to :

1. \( F_a(x,y) (t) = \varepsilon_0(t) \) and \( F_a(y,z) (t) = \varepsilon_0(t) \) then \( x = y \);
2. \( F_a(x,y) (t + s) \geq T(F_a(y,z) (t), F_a(x,y) (s)) \).

A Fuzzy Menger PQM-space is a Fuzzy Menger PM-space or Fuzzy Menger space if it satisfies the symmetry condition, i.e., \( F_a(y,z) (t) = F_a(x,y) (t) \) for all \( x, y, z \in X \).

The concept of neighborhoods in Fuzzy Menger space is introduced as

Definition

Let \( (X,F_a,T) \) be a Fuzzy Menger PQM-space. If \( x \in X, \varepsilon > 0 \) and \( \lambda \in (0,1) \), then \( (x,\lambda,\varepsilon) - \text{neighborhood of } x \), called \( U_{x,\varepsilon,\lambda} \), is defined by

\[ U_{x,\varepsilon,\lambda} = \{ y \in X : F_a(x,y) (\varepsilon) > (1 - \lambda) \} \]

An \((x,\lambda) - \text{topology in } X \) is the topology induced by the family \( \{ U_{x,\varepsilon,\lambda} : x \in X, \varepsilon > 0, \alpha \in [0,1] \} \) of neighborhoods.

Remark

If \( t \) is continuous, then Fuzzy Menger space \( (X,F_a,T) \) is a Hausdorff space in \((x,\lambda) - \text{topology}\).

Let \( (X,F_a,T) \) be a complete Fuzzy Menger PQM-space and \( A \subset X \). Then \( A \) is called a bounded set if

\[ \sup_{t > 0} \inf_{x,y \in A} F_a(x,y) (t) = 0 \]

Definition

(Schweizer and Sklar, 1983). A sequence \( \{ x_n \} \) in Fuzzy Menger PQM-space \( (X,F_a,T) \) is said to be convergent to a point \( x \in X \) if for every \( \varepsilon > 0 \) and \( \lambda > 0 \), there exists an integer \( N = N(x,\varepsilon,\lambda) \) such that \( x_n \in U_{x,\varepsilon,\lambda} \) for all \( n \geq N \) or equivalently \( F_a(x_n,x,\varepsilon) \geq 1 - \lambda \) for all \( n \geq N \) and \( \alpha \in [0,1] \).

Definition

(Schweizer and Sklar, 1983). A sequence \( \{ x_n \} \) in Fuzzy Menger PQM-space \( (X,F_a,T) \) is said to be Cauchy sequence if for every \( \varepsilon > 0 \) and \( \lambda > 0 \), there exists an integer \( N = N(x,\varepsilon,\lambda) \) such that \( F_a(x_n,x_m,\varepsilon) \geq 1 - \lambda \) for all \( n, m \geq N \).

If \( X \) itself is probabilistically bounded, then \( X \) is said to be a probabilistically bounded space.
Definition
(Schweizer and Sklar, 1983). A Fuzzy Menger QPM-space \((X, F_\alpha, T)\) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all \(\alpha \in [0,1]\).

Definition
(Hadžić and Pap, 2001). A t-norm \(T\) is of \(H\)-type (H-type in short) and \(T \in H\) if the family \(\{T^n\}_{n \in \mathbb{N}}\) of its iterates defined, for each \(x \in [0,1]\), by \(T^0(x) = 1, T^{n+1}(x) = T(T^n(x), x)\) for all \(n \geq 0\) is equicontinuous at \(x = 1\), that is \(\epsilon \in (0,1) \exists \delta(\epsilon) : x > 1 - \delta \Rightarrow T^n(x) > 1 - \epsilon\) for all \(n \geq 1\).

There is a nice characterization of continuous t-norm \(T\) of the class \(H\).

Definition
(Hadžić and Pap, 2001). If \(T\) is a t-norm and \((x_1, x_2, \ldots, x_n) \in [0,1]^n (n \in \mathbb{N})\), then \(T_{i=1}^n x_i\) is defined recurrently by \(T_{i=1}^1 x_i = 1\), if \(n = 0\) and \(T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)\) for all \(n \geq 1\). If \((x_n)_{n \in \mathbb{N}}\) is a sequence of numbers from \([0,1]\) then \(T_{i=1}^\infty x_i\) is defined as \(\lim_{n \to \infty} T_{i=1}^n x_i\) (this limit always exists) and \(T_{i=1}^\infty x_i\) as \(T_{i=1}^\infty x_{i+1}\).

In fixed point theory in probabilistic metric spaces there are of particular interest the t-norms \(T\) and sequences \((x_n) \in [0,1]\) such that \(\lim_{n \to \infty} T_{i=1}^n x_i = 1\) and \(\lim_{n \to \infty} T_{i=1}^n x_{i+1} = 1\).

Proposition
(Hadžić and Pap, 2001)
(1) If \(T \geq T_L\) then the following implication holds:
\[\lim_{n \to \infty} T_{i=1}^n x_{i+1} = 1 \iff \sum_{i=1}^{n} (1 - x_i) < \infty,\]

(2) If \(T \in H\) then for every sequence \((x_n)_{n \in \mathbb{N}}\) in \([0,1]\) such that \(\lim_{n \to \infty} x_n = 1\), one has \(\lim_{n \to \infty} T_{i=1}^n x_{n+1} = 1\).

Note that if \(T\) is a t-norm for which there exists \((x_n) \in [0,1]\) such that \(\lim_{n \to \infty} x_n = 1\) and \(\lim_{n \to \infty} T_{i=1}^n x_{n+1} = 1\), then \(\sup_{t \in [0,1]} T(t, t) = 1\).

Proposition
(Hadžić and Pap, 2001). Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of numbers from \([0,1]\) such that \(\lim_{n \to \infty} x_n = 1\) and \(T\)-norms \(T\) is of \(H\)-type then
\[\lim_{n \to \infty} T_{i=1}^n x_i = \lim_{n \to \infty} T_{i=1}^n x_{n+1} = 1.\]

Lemma
(Rezaian et al., 2008). If a Menger PQM-space \((X, F_\alpha, T)\) satisfies the following condition \(F_{\alpha(x,y)}(t) = C\), for all \(t > 0\) with fixed \(x, y \in X\). Then we have \(C = 1\) and \(x = y\).

Lemma
(Hadžić and Pap, 2001). Let the function \(\varphi(t)\) satisfy the following condition \((\varphi):\varphi^{-1}(0, \infty) \to [0, \infty)\) is non-decreasing and \(\sum_{n=1}^{\infty} \varphi^n(t) < \infty\) for all \(t > 0\), when \(\varphi^n(t)\) denotes the \(n^{th}\) iterative function of \(\varphi(t)\). Then \(\varphi(t) < t\) for all \(t < 0\).

Definition
(Singh and Jain, 2005). The mappings \(f, g : X \to X\) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is \(g(u) = f(u)\) for some \(u \in X\) then \(f(g(u)) = g(f(u))\).

MAIN RESULTS

Theorem
Let \((X, F_\alpha, T)\) a complete Fuzzy Menger PQM-space with continuous t-norm \(T\) of \(H\)-type and let \(f, g : X \to X\) be the mappings satisfying following conditions:
(i) \(g(X) \subseteq f(X)\);
(ii) Every convergent sequence in \(X\) has a unique limit;
(iii) \( F_{a_n}(x_{2n}) \) \( \geq \) \( F_{a_n}(x_{2n+1}) \), for all \( x, y \in X \) and \( t > 0 \) where the function \( \phi(t, x, y) : [0, 1] \rightarrow [0, 1] \) is onto, strictly increasing and satisfying conditions \( (\Phi) \).

(iv) \( f(X) \) is closed subset of \( X \).

(v) The pair \( (g, f) \) is weakly compatible.

(vi) Then \( f \) and \( g \) has coinciding point in \( X \) and has a unique common fixed point.

**Proof**

Let \( x_0 \) be an arbitrary point in \( X \). Since \( g(X) \subseteq f(X) \) there exist \( x_n \) such that \( f(x_n) = g(x_n) \). Inductively, we can construct sequences \( \{x_n\} \) and \( \{y_n\} \) such that \( y_{2n} = f(x_{2n+1}) = g(x_{2n}) \) for \( n \in \mathbb{N} \). Putting \( x = x_{2n} \) and \( y = x_{2n+1} \) in (iii) we have

\[
F_{a_n}(x_{2n+1}) \geq F_{a_n}(x_{2n}) \geq F_{a_n}(x_{2n-1}) \geq \ldots \geq F_{a_n}(x_0).
\]

Similarly, we can also prove that for all \( t > 0 \),

\[
F_{a_n}(y_{2n+1}) \geq F_{a_n}(y_{2n}) \geq F_{a_n}(y_{2n-1}) \geq \ldots \geq F_{a_n}(y_0).
\]

We show that \( \{y_n\} \) is a Cauchy sequence.

Let \( \varepsilon > 0 \) be given and \( \lambda \in (0, 1) \) be such that \( T^{x_n-1} 1 - \lambda \geq 1 - \lambda \). Also let \( t > 0 \) be such that \( F_{a_n}(f(x_n)) \geq T^{x_n-1} 1 - \lambda - \varepsilon \). Then, for every \( n \geq n_1 \) and \( m \in \mathbb{N} \) we have

\[
F_{a_n}(x_{2n+1}) \geq F_{a_n}(x_{2n}) \geq \ldots \geq F_{a_n}(x_0) \geq \Lambda_n - \varepsilon.
\]

Hence \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, \( \{y_n\} \) converges to \( z \) in \( X \). Thus

\[
\lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} f(x_{2n+1}) = \lim_{n \to \infty} g(x_{2n}) = z.
\]

Since \( f(X) \) is a closed subset of \( X \), there exists a point \( v \in X \) such that \( z = f(v) \).

Putting \( x = x_{2n} \) and \( y = v \) in (iii), we get

\[
F_{a_n}(x_{2n}) \geq F_{a_n}(x_{2n-1}) \geq F_{a_n}(x_{2n-2}) \geq \ldots \geq F_{a_n}(x_0).
\]

Now taking limit \( n \to \infty \), we have

\[
F_{a_n}(z_n) \geq F_{a_n}(z_{n-1}) \geq F_{a_n}(z_{n-2}) \geq \ldots \geq F_{a_n}(z_0) = 1.
\]

Hence, \( F_{a_n}(z_n) = 1 \Rightarrow g(v) = z \).

It shows that \( v \) is a coincidence point of \( f \) and \( g \).

Since the pair \( (g, f) \) is weakly compatible, we have \( g(f(v)) = F_{a_n}(v) \).

Hence \( g(z) = f(z) \).

Putting \( x = x_{2n} \) and \( y = z \) in (iii), we get

\[
F_{a_n}(x_{2n+1}) \geq F_{a_n}(x_{2n}) \geq F_{a_n}(x_{2n-1}) \geq \ldots \geq F_{a_n}(x_0).
\]

On the other hand, since \( F_{a_n} \) is non-decreasing, we get

\[
F_{a_n}(z_n) \geq F_{a_n}(z_{n-1}) \geq F_{a_n}(z_{n-2}) \geq \ldots \geq F_{a_n}(z_0) = 1.
\]

Hence \( F_{a_n}(z_n) = 1 \) for all \( n > 0 \). From Lemma 2.1 we conclude that \( C = 1 \), that is \( g(z) = z \).

Now combine all the results, we get \( g(z) = f(z) = z \).

Which implies \( z \) is a common fixed point of \( f \) and \( g \) in \( X \).

**Uniqueness**

Let \( w \neq z \) be another common fixed point of \( f \) and \( g \).

Taking \( \bar{x} = x \) and \( \bar{y} = w \) in (iii), we have

\[
F_{a_n}(x_{2n}) \geq F_{a_n}(x_{2n-1}) \geq F_{a_n}(x_{2n-2}) \geq \ldots \geq F_{a_n}(x_0).
\]

Since \( F_{a_n} \) is non-decreasing, we get

\[
F_{a_n}(z_{n+1}) \leq F_{a_n}(z_n) \leq F_{a_n}(z_{n-1}) \leq \ldots \leq F_{a_n}(z_0) = 1.
\]

Hence \( F_{a_n}(z_{n+1}) = C \) for all \( n > 0 \). From Lemma 2.1 we conclude that \( C = 1 \), that is \( z = w \) and so the uniqueness of the common fixed point.
Now we extend our result to finite number of mappings in Menger PQM-space.

**Theorem**

Let \((X, \mathcal{F}_\alpha, T)\) a complete Fuzzy Menger PQM-space with continuous \(t\)-norm \(T\) of H-type and let \(f_1, f_2, \ldots, f_n : X \to X\) be the mappings satisfying following conditions:

(i) \(g(X) \subseteq f_1 f_2 \ldots f_n(X)\);

Every convergent sequence in \(X\) has a unique limit;

(ii) \(F_a((x, y)) \geq F_a(f_1 f_2 \ldots f_n X, x, y)\) for all \(x, y \in X\) and \(t > 0\) where the function \(g(x) : [0, 1] \to [0, 1]\) is onto, strictly increasing and satisfying conditions \((\Phi)\);

(iii) \(f_1 f_2 \ldots f_n(X)\) is a closed subset of \(X\).

(iv) \(g(f_2 \ldots f_n(X)) = f_2 \ldots f_n g(X)\);

\(g(f_1 \ldots f_n) = f_1 \ldots f_n g(X)\);

\(f_1 f_2 \ldots f_n = f_n f_{n-1} \ldots f_2 f_1\);

\(f_1 \ldots f_{n-1} f_n = f_n f_{n-1} \ldots f_2 f_1 f_2 \ldots f_n\);

(v) The pair \((g, f_1 f_2 \ldots f_n)\) is weakly compatible.

Then \(f_1 f_2 \ldots f_n\) and \(g\) has coinciding point in \(X\) and has a unique common fixed point.

**Proof**

If we put \(f_1 f_2 \ldots f_n = t\) in Theorem 3.1 then we get \(g(t) = t f_1 f_2 \ldots f_n g(x) = z\).

Now we show that \(z\) is a common fixed point of all the component mappings, by putting \(x = z\), \(y = f_2 \ldots f_n z\) and \(f_1 = f_1 f_2 \ldots f_n\) in (iii), we get

\(F_a(z, z) \geq F_a(z, f_2 \ldots f_n z)\) or \(F_a(z, z) \geq F_a(z, f_1 f_2 \ldots f_n z)\).

Since \(F_a\) is non-decreasing, we get

\(F_a(z, z) \geq F_a(z, z)\).

Hence \(F_a(z, z) = C\) for all \(t > 0\). From Lemma 2.1 we conclude that \(C = 1\), that is \(f_1 f_2 \ldots f_n z = z\). Thus, \(f_1 z = f_1 f_2 \ldots f_n z = \ldots = f_n z = z\).

Similarly, we have \(f_1 z = f_2 z = f_3 z = \ldots = f_n z = z\).

So there exists a common fixed point \(z \in X\) such that \(f_1 z = f_2 z = f_3 z = \ldots = f_n z = z = g(z)\).

Uniqueness of the common fixed point follows easily from (iii).

If the condition \(T\) is of H-type is replaced by

\(\lim_{n \to \infty} T^{\frac{1}{n}} = F_a((x, g(x)) = 1\)

and

\(\lim_{n \to \infty} T^{\frac{1}{n}} = F_a((y, g(x)) = 1\) for some \(\mu \in (0, 1)\).

Taking into account Proposition (2.1), we get the following corollaries:

**Corollary**

Let \((X, \mathcal{F}_\alpha, T_L)\) be a complete Fuzzy Menger PQM-space. Let \(f_1 f_2 \ldots f_n : X \to X\) be the mappings satisfying following conditions:

(i) \(g(X) \subseteq f_1 f_2 \ldots f_n(X)\);

(ii) Every convergent sequence in \(X\) has a unique limit;

(iii) \(F_a((x, y)) \geq F_a(f_1 f_2 \ldots f_n X, x, y)\) for all \(x, y \in X\) and \(t > 0\) where the function \(g(x) : [0, 1] \to [0, 1]\) is onto, strictly increasing and satisfying conditions \((\Phi)\);

(iv) \(f(X)\) is a closed subset of \(X\).

(v) The pair \((g, f_1 f_2 \ldots f_n)\) is weakly compatible.

Then \(f\) and \(g\) has coinciding point in \(X\) and has a unique common fixed point provided that

\[\sum_{i=1}^{\infty} \left(1 - F_a((x, g(x)) = \frac{1}{\mu})\right) < \infty\]

for some \(x \in X\) and some \(\mu \in (0, 1)\).

**Corollary**

Let \((X, \mathcal{F}_\alpha, T_L)\) be a complete Menger PQM-space. Let \(f_1 f_2 \ldots f_n : X \to X\) be the mappings satisfying following conditions:

(i) \(g(X) \subseteq f_1 f_2 \ldots f_n(X)\);

(ii) Every convergent sequence in \(X\) has a unique limit;
Remark

The conclusions of Theorem 3.1, Theorem 3.2, Corollary 3.1 and Corollary 3.2 remain true for \( \varphi(t) = kt \), where \( k \in (0,1) \).

REFERENCES


